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Solitary burn-up waves in a multiplying medium

A fast reactor in which a solitary wave of neutrons, stabilized by the burn-up process, propagates autocatalytically through fertile material like U-238 or Th-232 is a completely new idea in reactor technology. Starting with the stationary monoenergetic neutron diffusion equation in one-dimensional geometry the main properties of the burn-up wave are derived and shown that in deed a fast reactor, possessing fuel elements of only fertile material, can be possible.

Solitäre Abbrandwellen in einem multiplizierenden Medium. *Ein schneller Reaktor, in welchem sich eine Abbrandwelle autokatalytisch durch fruchtbares Material wie U-238 oder Th-232 „durchbrennt" ist eine vollkommen neue Idee in der Reaktortechnik. Ausgehend von der stationären ein-dimensionalen Diffusionsgleichung werden die Charakteristika der Abbrandwelle abgeleitet und es wird dargelegt, dass ein schneller Reaktor, dessen Brennelemente nur aus fruchtbarem Material besteht, in den Bereich des konzeptionell Möglichen rückt.*

1 Introduction

Recent research work [1–4] indicates that it might be possible to design a completely new fast reactor. Once "ignited" by a strong neutron source or a small critical zone a solitary neutron wave, i. e, a burn-up wave, propagates autocatalytically through fertile material. Inside the wave the fertile material is converted partially into bred fissile material which is fissioned and converted into fission products thereby producing energy. In contradistinction to a classical fission reactor the zone in which the chain reaction takes place propagates slowly through the reactor – thus opening a new perspective in the design of a new reactor type with a spatially non-stationary reactor core.

2 Basic equations in infinite geometry

Consider the possibility of an infinite U-238 medium. If any kind of neutron source, for instance a critical reactor zone, is installed at one end of the U-238 block neutrons will penetrate into this fertile material converting it through capture reactions successively into fissile material which will be fissioned afterwards releasing thereby new neutrons supporting this penetration process.

If, after some time in equilibrium, a self-sustained neutron wave along the x-axis develops, the following neutron balance holds:

$$D\cdot\Phi_{xx}(x,t) + [(\eta^{fis}-1)\cdot\sigma_a^{fis}\cdot N_{fis}(x,t) - (1-\eta^{fer})\cdot\sigma_a^{fer}\cdot N_{fer}(x,t) - \Sigma_a^s]\cdot\Phi(x,t) = 0 \quad (1)$$

The notation is self-explanatory. The diffusion constant D is considered to be constant and Σ_a^s represents a constant macroscopic absorption cross section due, for example, to structural materials. The fertile material ("fer") is U-238 and the fissile material produced ("fis") is Pu-239. The absorption of the fission products are neglected as well as the built-up of higher isotopes. The assumptions are kept as simple as possible to explain the wave character of Eq. (1) if burn-up proceeds.

Opposite to earlier considerations, the RHS of Eq. (1) is set to be zero since the wave character of Eq. (1) should develop endogenously as proposed by Van Dam [5, 6].

In order to convert Eq. (1) into a wave equation for the (dimensionless) burn-up or fluence function $F(x-vt)$, where v is the propagation velocity, we introduce

$$\sigma_a^{fer}\cdot\int_0^t \Phi(x,\tau)\cdot d\tau = F\left[\frac{x-vt}{L_0}\right] \quad (2)$$

where $1/L_0$ is the wave-number and L_0 is the diffusion length in the original fertile material ($L_0^2 = D/\Sigma_a^{fer,0}$) and we obtain in equilibrium

$$\Phi = \frac{1}{\sigma_a^{fer}}F_t = \frac{-v}{L_0\cdot\sigma_a^{fer}}\dot{F}$$

$$\Phi_x = \frac{-v}{L_0^2\cdot\sigma_a^{fer}}\ddot{F}, \qquad \Phi_{xx} = \frac{-v}{L_0^3\cdot\sigma_a^{fer}}\dddot{F}$$

$$\Phi_t = \frac{v^2}{L_0^2\cdot\sigma_a^{fer}}\ddot{F}, \qquad \Phi_{tt} = \frac{-v^3}{L_0^3\cdot\sigma_a^{fer}}\dddot{F} \quad (3)$$

where a dot means the derivative to the whole (dimensionless), argument $z=(x-vt)/L_0$. Eq. (3) also fulfills the wave equation for the neutron flux being $\Phi_{tt} - v^2\Phi_{xx} = 0$.

Since the propagation velocity is very small [3] the burn-up equations can be written in the *prompt* approximation as function of F by

$$N_{fer}(x,t) = N_{fer,0}\cdot e^{-F}$$

$$N_{fis}(x,t) = \frac{\sigma_c^{fer}}{\sigma_a^{fis}-\sigma_a^{fer}}\cdot N_{fer,0}\cdot\left[e^{-F}-e^{-\alpha F}\right] \quad (4)$$

with $N_{fis}(F=0)=0$ and $N_{fer}(F=0)=N_{fer,0}$ (deep in the fertile material).

Introducing now the abbreviations

$$\alpha = \sigma_a^{fis}/\sigma_a^{fer}, \quad \beta = \sigma_c^{fer}/\sigma_a^{fer}, \quad a = \Sigma_a^s/\Sigma_a^{fer,0}$$

and using Eqs. (2–4), the PDE of Eq. (1) can be transformed into an autonomous ODE of 3rd order for F(z) being

$$\dddot{F} + Z(F)\cdot\dot{F} = 0 \quad (5)$$

with the (dimensionless) buckling

$$Z(F;a) = \frac{\alpha\,\beta}{\alpha-1}(\eta^{fis}-1)\cdot\left[e^{-F}-e^{-\alpha F}\right] + (\eta^{fer}-1)\cdot e^{-F} - a \quad (6)$$

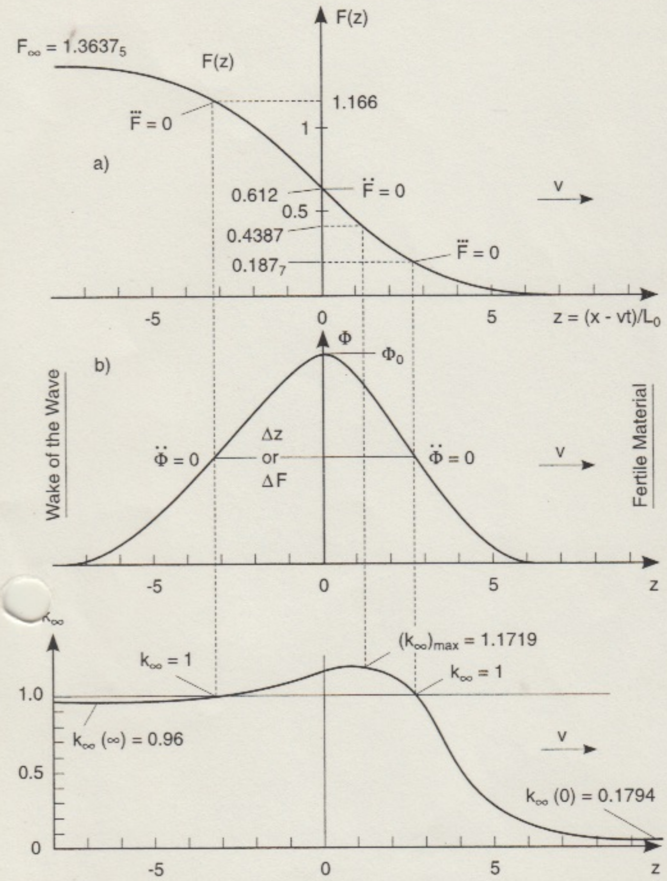

Fig. 1. (a) Fluence or burn-up F, (b) Neutron flux Φ and (c) k_∞ as function of $z = (x - vt)/L_0$

where a is the parasitic absorption parameter which can be chosen arbitrarily and which will be adjusted later on.

To reduce the order of Eq. (5), an integration is carried out directly under the boundary condition that $\ddot{F}(F = 0) = 0$ (i. e, the flux-gradient deep in the fertile material is zero) yielding

$$U(F;a) = (\eta^{fis} - 1)\frac{\beta}{\alpha - 1}\left[\alpha e^{-F} - e^{-\alpha F}\right] + (\eta^{fer} - 1)\cdot e^{-F} + aF - (\beta\,\eta^{fis} + \eta^{fer} - \beta - 1) \qquad (7)$$

Multiplying both sides with $\dot{F}$ and integrating it again under the condition that $\dot{F}(F = 0) = 0$ (i. e, the flux deep in the fertile material is zero) we get finally an autonomous ODE of first order being

$$\dot{F} = V(F;a) = +\sqrt{2}\cdot\left[\frac{\beta(\eta^{fis} - 1)}{\alpha(\alpha - 1)}\left\{\alpha^2(1 - e^{-F}) + e^{-\alpha F} - 1\right\} - (1 - \eta^{fer})\cdot(1 - e^{-F}) + \frac{a}{2}\cdot F^2 - \beta\,\eta^{fis} + \eta^{fer} - \beta - 1)\cdot F\right]^{0.5} \qquad (8)$$

where the RHS possesses two ground states (zeros): $F = 0$ and $F = F_\infty$.

Furthermore, the numerical choice of the parasitic absorption term, a, allows to adjust the criticality of the irradiated fuel in the "wake" of the wave. The latter possesses the maximum and asymptotic burn-up F_∞ and its criticality can be expressed by the infinite multiplication constant $k_\infty(\infty)$.

If , for instance, $k_\infty(\infty) \geq 1$, the irradiated fuel of the wake has to be removed from time to time or it has to be poisoned by inserting additional absorber rods in order to keep the whole system subcritical in the long run.

If, however, $k_\infty(\infty) < 1$, the system works without any external human control or in a "self-relianced" manner. In the latter case $Z(F_\infty;a)$ of Eq. (5) has to be negative.

To adjust the parameter a accordingly we request in Eq. (5) that $\dddot{F}(F_\infty) = 0$ (i.e., the curvature of the neutron flux is zero) if $Z(F_\infty;a) = k_\infty(\infty) - 1$. $\dddot{F} = 0$ in Eq. (5) is fulfilled if either $Z(F_\infty;a) = 0$ or if $V(F_\infty;a) = 0$ in Eq. (8), delivering the two conditions

$$Z(F_\infty;a) = k_\infty(\infty) - 1$$

$$V(F_\infty;a) = 0 \qquad (9)$$

For a given $k_\infty(\infty)$ the asymptotic burn-up F_∞ and a can be determined numerically by Eq. (9).

2 Numerical evaluation

Introducing the data of Table 1 for a typical Na-cooled fast system into Eq. (9) and requesting, for example, that $k_\infty(\infty) = 0.96$ (about – 10$ subcritical) we obtain

$$F_\infty = 1.36375 \text{ and } a = 0.2246 \qquad (10)$$

For comparison, for $k_\infty(\infty) = 1$ we would obtain $F_\infty = 1.20388$ and $a = 0.21636$.

Introducing $a = 0.2246$ into Eq. (8) or Eq. (7) and integrating it numerically using a Runge-Kutta scheme on the TI-92 calculator we obtain the kink of F(z) shown in Fig. 1 (a) with its characteristic jump from $F = 0$ to $F = F_\infty = 1.36375$. The zero point of the z-axis was chosen to be at the turning point of F, ($\ddot{F} = 0$), at $F = 0.612$.

Fig. 1 (b) shows the neutron flux as function of z according to Eq. (3) with the flux peak denoted by Φ_0. Its width is $\Delta F = 1.166 - 0.1877 = 0.9783$ or about 5.6 L_0 or about 68 cm.

Fig. 1 (c) shows $k_\infty(z)$ with $k_\infty(\infty) = 0.96$ and $k_\infty(0) = 1 + 0.404 - 1 = 0.1794$. k_∞ is 1 at the turning points of Φ. Inside these two points $k_\infty > 1$ with a maximum of 1.1719 if $F = 0.4387$.

The RHS of Eq. (8), V(F;a) for $a = 0.2246$, is shown in Fig. 2. The ground states (zeros) are $F = 0$ and $F = F_\infty = 1.36375$ and V is slightly asymmetrical around $F_\infty/2$. This is the reason why Φ(z) in Fig. 1 (b) is not an even function with respect to $z = 0$. The shape of $V(F; a = 0.2246)$ resembles to a Cassini's oval, to a lemniscate of Jakob Bernoulli or to the

Table 1. Fast monoenergetic nuclear data used [4]

U-238 – Fertile Material

$\sigma_{tr} = 8.2\,b$, $\sigma_c^{fer} = 0.293\,b$, $\sigma_f^{fer} = 0.05b$, $\sigma_a^{fer} = 0.345\,b$,
$\eta^{fer} = \bar{\nu}\,\sigma_f^{fer}/\sigma_a^{fer} = 0.404$
$N_{fer,0} = 2.81 \cdot 10^{22}/cm^3$ (=60 % of metallic U density)
$D = 1/(3 \cdot \Sigma_{tr}^{fer,0}) = 1.45\,cm$, $L_0 = (D/\Sigma_a^{fer,0})^{.5} = 12.2\,cm$

Pu-239 – Fissile Material

$\sigma_a^{fis} = 2.32b$, $\eta^{fis} = \bar{\nu}\sigma_f^{fis}/\sigma_a^{fis} = 2.32$, $\sigma_f^{fis} = 1.82\,b$

Dimensionless Abbrevations

$\alpha = \sigma_a^{fis}/\sigma_a^{fer} = 6.71$, $\beta = \sigma_c^{fer}/\sigma_a^{fer} = 0.85$
$\gamma = \sigma_f^{fis}/\sigma_a^{fer} = 5.275$, $\delta = \sigma_f^{fer}/\sigma_a^{fer} = 0.145$
$a = \Sigma_a^s/\Sigma_a^{fer,0}$

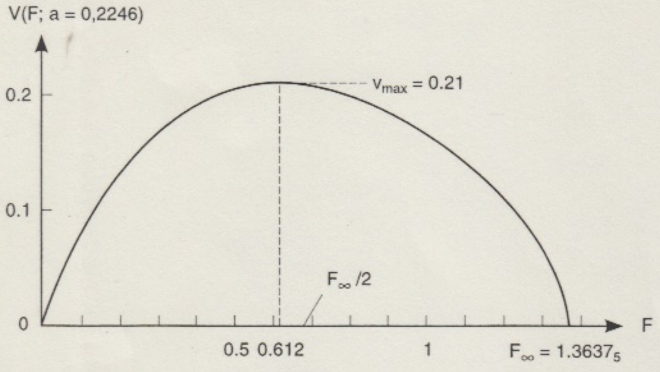

Fig. 2. V(F; a) for a = 0.2246 of Eq. (8) as function of F; the two ground states (zeros) are F = 0 and F = F∞ = 1.36375

first of the 12 Jacobian elliptic functions rather than to a simple logistic product or to a sine-function. We therefore dispense us here with trying to find good approximations for an analytical representation.

3 Other important characteristics

3.1 The wave velocity, v

According to the first equation of Eqs. (3) the absolute value of the wave velocity, |v|, is given by

$$|v| = L_0 \cdot \sigma_a^{fer} \cdot \frac{\Phi_0}{(F)_{max}} = \frac{L_0 \cdot \sigma_a^{fer} \cdot \Phi_0}{V_{max}} \tag{11}$$

being proportional to the peak flux Φ_0 and being inversely proportional to the original atomic particle density of the fertile material, $N_{fer,0}$ (or to its density).

Assuming, for instance, $\Phi_0 = 10^{15}$ n/(cm² s) we obtain with $V_{max} = 0.21$ from Fig. 2

$$|v| = 2 \cdot 10^{-8}\ \text{cm/s} \approx 6.3\ \text{mm/year} \tag{12}$$

Theoretically Φ_0 depends on the initial boundary condition of the igniting process, in practice, however, Φ_0 can be chosen depending on the power coefficient of the reactivity of the system.

3.2 The wake properties

The relatively high asymptotic burn-up of F = 1.36275 (due to the simplified assumptions we made in Eq. (1)) means that the irradiated fuel consists essentially of fission products. The fertile material reduces to $N_{fer,\infty}/N_{fer,0} = 26\,\%$ of its initial value.

The asymptotic enrichment ε in the wake is, in accordance with Eq. (4), given by

$$\varepsilon = \frac{N_{fis,\infty}}{N_{fer,0}} = \frac{\beta}{\alpha - 1}\left[1 - e^{-(\alpha+1)F_\infty}\right]$$

$$\approx \frac{\beta}{\alpha - 1} \approx 14.9\,\%. \tag{13}$$

ε varies monotonically from ε = 0 (F = 0, deep in the fertile material) up to the above value in the wake of the wave ($F = F_\infty$).

3.3 The specific power density, p

The spatial specific power density distribution is defined by

$$p = \left[\sigma_f^{fis} N_{fis} + \sigma_f^{fer} N_{fer}\right] \cdot \Phi \qquad \text{in fissions/(cm}^3\text{ s)} \tag{14}$$

which can be written by means of Eqs. (3), (4), (8) and (11) and by introduction of two further abbreviations

$$\gamma = \sigma_f^{fis}/\sigma_a^{fer}, \qquad \delta = \sigma_f^{fis}/\sigma_a^{fer}$$

in the form

$$p(F) = \Sigma_a^{fer,0} \cdot \Phi_0 \cdot \left[\frac{\gamma\beta}{\alpha-1}(e^{-F} - e^{-\alpha F}) + \delta \cdot e^{-F}\right] \cdot \frac{V(F;a)}{V_{max}}$$

$$= \Sigma_a^{fer,0} \cdot \Phi_0 \cdot G(F) \tag{15}$$

With a = 0.2246 of the above case G varies from zero (= G(0), deep in the fertile material) to 0.3281 (= G(0.1877), at the first turning point of Φ) to 0.4916 (= G(0.612), where the flux peaks) to 0.1683 (= G(1.166), at the second turning point of Φ) to practically zero (= $G(F_\infty)$, in the wake). p peaks at F = 0.4565 (slightly right of the flux peak) with $G_{max} = 0.5285$.

With $N_{fer,0} = 2.81 \cdot 10^{22}$/cm³ and $\Phi_0 = 10^{15}$ n/(cm³s) the maximum power density, p_{max}, is therefore ≈ 164 MW_{th}/m^3.

3.4 The role of k_{eff}

The classical definition of the effective multiplication constant in a monoenergetic, geometrically finite, system is

$$k_{eff} = \frac{k_\infty}{1 + L_0^2 \cdot B^2} \tag{16}$$

where B² is the (geometric) buckling and where k_∞ is constant. Opposite to a spatially stationary finite critical mass we do not have the *Dirichelet* condition (Φ = 0 on the boundaries), instead we have in our case the *Cauchy* conditions (Φ and $\Phi_x = 0$ on the boundaries, see Fig. 1 (b)), meaning that the leakage current j = −D grad Φ is zero resulting in an excellent neutron economy.

According to Eq. (9) $k_\infty = 1 + L_0^2 B^2 = 1 + Z(F;a)$ and, therefore, we formally obtain

$$k_{eff} = 1 \tag{17}$$

as expected heuristically.

4 Conclusions

It was shown that the time-stationary diffusion equation possesses a wave solution, controlled by the burn-up process. The wave propagates autocatalytically into fuel elements containing exclusively fertile material. By means of the amount of an initially added parasitic absorber (e.g. structural material) the reactivity of the irradiated fuel can be pre-determined.

In a practical reactor configuration one would use a cylinder of fertile material which will be ignited in the middle and two solitary burn-up waves would finally propagate in opposite axial directions as proposed for the first time by *E. Teller* et al. [1]. The proposal of the author is to design a ring-reactor in the form of a torus (like a fusion reactor) in which several burn-up waves can propagate around – one behind the other. Once ignited, or once loaded according to Fig. 1 (c), one would have a fast breeder system without any reprocessing and Pu transports. At the end of its useful lifetime the site of the reactor could simultaneously be the site of the radioactive waste disposal if it is being built underground in a suitable surrounding (e. g. in a sand-bed).

The model presented is the simplest way to illustrate this novel possibility to design a fast nuclear reactor possessing a spatially non-stationary critical mass. Such a reactor does not work only with the U-238/Pu cycle but also with the Th-232/U-233 cycle.

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Appendix

For the sake of completeness it should be noted that one could also start with the PDE of Eq. (1) to obtain an ODE for Φ. Expressing $N_{fis}(x,t)$ and $N_{fer}(x,t)$ by Eq. (4) with $\sigma_a^{fer} \cdot \int_0^t \Phi(x,\tau)d\tau$ and dividing it by $\Phi(x,t)$, differentiating it with respect to t, one can eliminate the e^{-F} and $e^{-\alpha F}$ terms by repeating this procedure twice. The following non-linear PDE of 4th order is obtained:

$$\begin{aligned}-L_0^2 \cdot \Phi^2 \cdot \Phi_{xxtt} = {} & \alpha \cdot \sigma_a^{fer} \cdot (a\,\Phi + L_0^2 \cdot \Phi_{xx}) \cdot \Phi^4 \\ & + L_0^2 \cdot (\alpha+1) \cdot \sigma_a^{fer} \cdot (\Phi\,\Phi_{xxt} - \Phi_t\,\Phi_{xx}) \cdot \Phi^2 \\ & - L_0^2 \cdot \left[(3\,\Phi_t \cdot \Phi_{xxt} + \Phi_{tt} \cdot \Phi_{xx}) \cdot \Phi - 3\,\Phi_t^2 \cdot \Phi_{xx}\right] \end{aligned} \quad \text{(A.1)}$$

This PDE for the space- and time-dependent neutron flux $\Phi(x,t)$ cannot be solved by separating space and time. However, if we make the wave approach

$$\Phi(x,t) = \frac{v}{L_0 \cdot \sigma_a^{fer}} \cdot \Psi\left[\frac{x - vt}{L_0}\right] \quad \text{(A.2)}$$

with $z = (x - vt)/L_0$ and the amplitude proportional to the propagation velocity, v, we obtain a highly non-linear ODE of 4th order for the (dimensionless) flux, depending on the (dimensionless) variable z being

$$\begin{aligned}-\Psi^2 \cdot \ddddot{\Psi} = {} & \alpha\,a\,\Psi^5 + \alpha\,\ddot{\Psi} \cdot \Psi^4 - (\alpha+1) \cdot \dddot{\Psi} \cdot \Psi^3 \\ & + (\alpha+1)\,\dot{\Psi} \cdot \ddot{\Psi} \cdot \Psi^2 - (3\,\dot{\Psi} \cdot \dddot{\Psi} + \ddot{\Psi}^2)\,\Psi + 3\,\dot{\Psi}^2 \cdot \ddot{\Psi} \end{aligned} \quad \text{(A.3)}$$

with $\dot{\Psi} = d\Psi/dz$ etc. Eq. (A.3) is a non-homogeneous biharmonic ODE containing only the parameters a and α. It can be solved numerically with the initial conditions $\Psi(-\infty) \ldots \ddddot{\Psi} = (-\infty) = 0$ yielding $\Psi(z)$. It possesses the shape of Fig. 1 (b).

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